Year 13 Mathematics

IAS 3.2

Linear Programming

Robert Lakeland & Carl Nugent

Contents

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NCEA 3 External Achievement Standard 3.2 – Apply Linear Programming Methods in Solving Problems.

This achievement standard involves applying linear programming methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply linear programming methods in solving problems.	• Apply linear programming methods, using relational thinking, in solving problems.	• Apply linear programming methods, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objective:
  - Use linear programming techniques.
- Apply linear programming methods in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.
- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate a situation
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.
- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or statistical contexts.
- Methods include a selection from those related to:
  - linear inequalities
  - feasible regions
  - optimisation.

Graphing Linear Equations

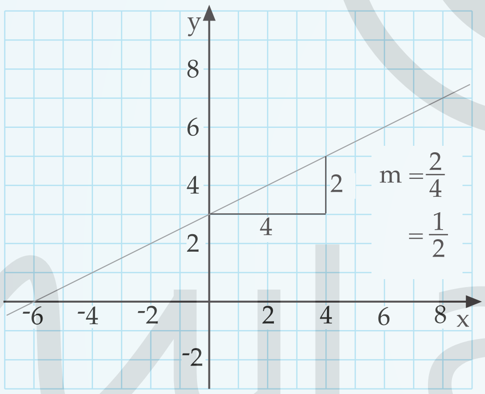
Linear Equations

In order to graph linear inequalities such as $y \leq 12 - 2x$, we need to first graph a linear equation (in this case $y = 12 - 2x$) and then shade one side of it to represent the inequality.

A linear equation is typically expressed in two different forms and we need to revise how we graph both forms.

Equations in the form y = mx + c

In Year 10 / 11 we learnt that an equation in the form $y = mx + c$ is a straight line which crosses the y-axis at (0, c) with a gradient of m.



For example the line $y = 0.5x + 3$ is drawn through (0, 3) with a gradient of 0.5.

Equations in the form ax + by = k

In linear programming the linear expressions are usually not expressed in the form $y = mx + c$ but are often written in the form

$$ax + by = k$$

We can manipulate this and any other forms to the form $y = mx + c$ using algebra

$$\begin{aligned} ax + by &= k \\ by &= {}^-ax + k \\ y &= \frac{^-a}{b}x + \frac{k}{b} \end{aligned}$$

where $m = \frac{^-a}{b}$

and $c = \frac{k}{b}$

We need to do this if we are using our graphics calculator to graph the line.

Extra Notes

Optimising a Solution

Finding an Optimum Solution

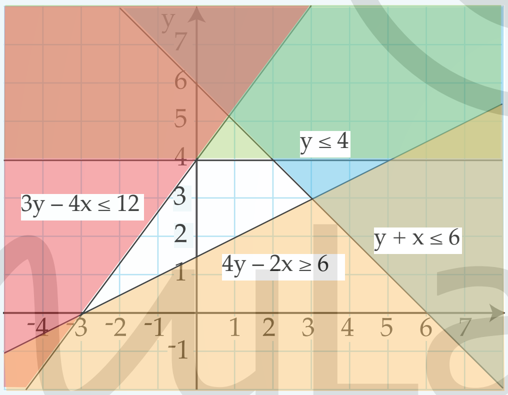
When an area is identified which fulfils the set conditions, the best or optimum solution from within this area still has to be identified.

All points in the identified area meet the minimum conditions so a given function is used to find which of these points is the best or optimum solution.

The inequalities

$$3y - 4x \leq 12$$
$$y + x \leq 6$$
$$4y - 2x \geq 6$$
$$y \leq 4$$

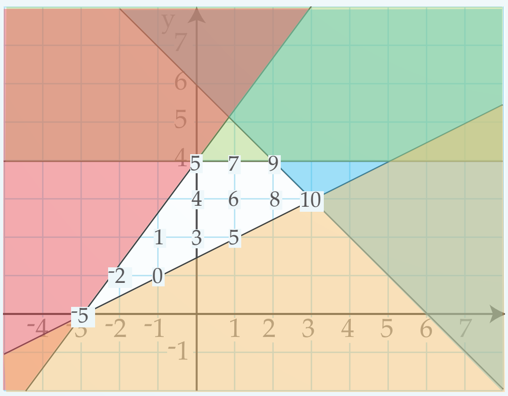
have been graphed below.



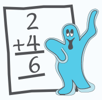
Every point not shaded meets the conditions of all 4 inequalities so to find the best or optimum condition we need additional information. If, for example, we were asked to find the point which optimised the function

$$F = 2x + y + 1$$

we could test every coordinate in the range. In the graph below we have evaluated F at every point that is in the region that fulfils the initial conditions. We can see that F varies from $^-5$ (minimum) to 10 (maximum).



Extra Notes

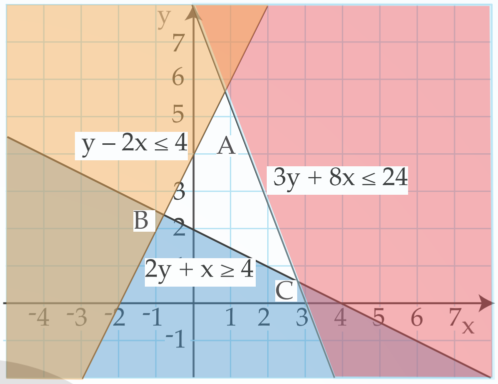


Example

Find the coordinates of any point that fulfils the inequality and gives the largest value for the function

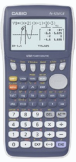
$$F = 2y + 3x$$

This question allows non-integer solutions which cannot be read from the graph so we use simultaneous equations to find the intersection. This can be done on a graphics calculator.



To solve simultaneous equations on your calculator every equation must be in the form ax + by = c. We write each inequality as an equation and rearrange them into the form ax + by = c. We find the intersections by using the equation of the lines that intersect at each vertex.

$$^{-}2x + y = 4 \qquad (1)$$
$$8x + 3y = 24 \qquad (2)$$
$$x + 2y = 4 \qquad (3)$$



Using your Casio 9750GII to find the coordinates of point A.

$$^{-}2x + y = 4 \qquad (1)$$

and $$8x + 3y = 24 \qquad (2)$$

1. Select the EQUA mode from the menu and select simultaneous equations.

   [MENU] [8] (EQUA) [F1] (SIML)

2. For 2 unknowns (2 equations) select F1.

   [F1] (2)

3. Enter the two equations.

   [(–)] [2] [EXE] [1] [EXE] [4] [EXE]

   [8] [EXE] [3] [EXE] [2] [4] [EXE]

4. Solve the simultaneous equations.

   [F1] (SOLV) OR [EXE]

5. The answer is x = 0.8571 and y = 5.714

To find B and C we use the pair of equations that pass through each point. B is the point ($^{-}$0.8, 2.4).

Using the TI-84 Plus to find the coordinates of C.

To solve

$$x + 2y = 4 \qquad (1)$$
$$8x + 3y = 24 \qquad (2)$$

1. Select applications menu and scroll down to PlySmlt2

   [APPS] [▼] . . . . [▼] [▼] [ENTER] [ENTER]

2. For Simult Eqn Solver and 2 unknowns and 2 equations select

   [2] [▼] [▼] [GRAPH] (NEXT)

3. Enter the two equations.

   [1] [ENTER] [2] [ENTER] [4] [ENTER]

   [8] [ENTER] [3] [ENTER] [2] [4] [ENTER]

4. Solve the simultaneous equations with F5.

   [GRAPH] (F5)

5. The answer is x = 2.769 and y = 0.6154

Point A is (0.8571, 5.714) so

$$F = 2 \times 5.714 + 3 \times 0.8571$$
$$= 14.00 \qquad (4 \text{ sf})$$

Point **B** is ($^{-}$0.8, 2.4) so

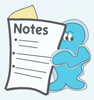
$$F = 2 \times 2.4 + 3 \times {}^{-}0.8$$
$$= 2.4$$

Point **C** is (2.769, 0.6154) so

$$F = 2 \times 0.6154 + 3 \times 2.769$$
$$= 9.538 \qquad (4 \text{ sf})$$

The best solution is the point A (0.8571, 5.714) which gives F as 14.00 (4 sf).

Linear Programming Application Using Desmos



Graphing Tools — Desmos

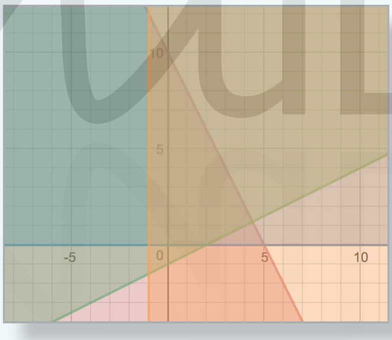
There are many graphing programmes that can assist with linear programming.

We will demonstrate two in this workbook. One is an advanced graphing calculator (Desmos) and the second is a stand alone linear programming grapher (LP Grapher). If you are already comfortably using a different programme then the authors recommend you continue to use it.

The advantage of Desmos is it is free and available for all devices and online. You can work online using their website or download Desmos as an app on an Apple iPad or an Android tablet. In addition to graphing equations, Desmos will also graph and shade inequalities.

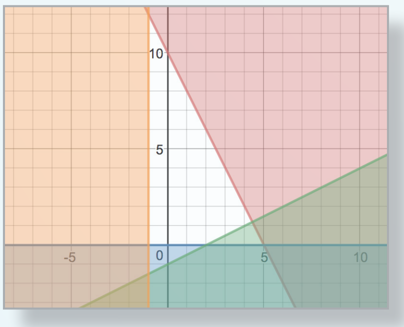
What Desmos does not do is shade out. It shades the area described by the inequality. For example if we had the set of inequalities

$$y + 2x \leq 10$$
$$2y - x \geq {}^{-}2$$
$$y \geq 0$$
$$x \geq {}^{-}1$$



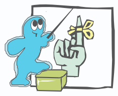
The polygon that meets our conditions is obscured by overlapping shading.

The area we want is obscured by the overlapping shading. To shade out we **reverse** the direction of all the inequalities.



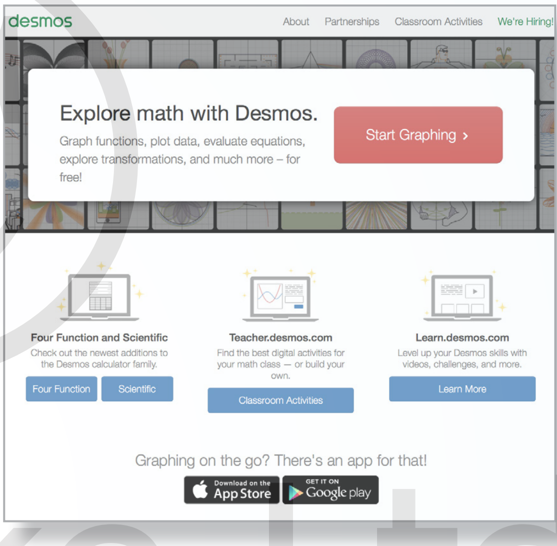
When we shade out the polygon that meets our conditions is now white and we can identify the vertices for checking.

You can choose to create an account so that you can save graphs and access them later. Then you can start graphing a problem on your iPad and load it and complete it from a different device later.

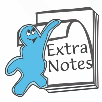


Desmos runs anywhere on any device. It promises no advertisements and no student data is sold or shared with third parties. It is an educational tool used by millions of students and paid for by publishers and educational organisations.

It was funded by Kapor Capital, Learn Capital, Kindler Capital, Elm Street Ventures and Google Ventures and will always be free to use.



You can access the website at www.desmos.com and select Start Graphing or click on the links from where you can download an application for your tablet. The programme does a lot more than graph inequalities and it has links to videos explaining what it can do.



Example

Use the Desmos graphing programming and shading out to solve the following.

An iron smelter has two sources of iron ore, mine Alberta and mine Buffalo.

- In order to keep either plant running, at least twelve tonnes of iron ore must be processed each day.
- Ore from mine Alberta costs $40 per tonne and ore from mine Buffalo costs $20 per tonne. Iron ore costs must be kept to less than $600 per day.
- Government regulations require that the amount of ore from mine Alberta must be less than twice the amount of ore from mine Buffalo.
- Both mines are limited to a maximum of 16 tonnes per day.

Ore from mine Alberta yields 100 kg of iron per tonne, and ore from mine Buffalo yields 150 kg of iron per tonne. How many tonnes of ore from both mines must be processed each day to maximize the amount of iron smelted subject to the above constraints?

First express the conditions as inequalities. Let mine production from Alberta be x and from Buffalo be y.

"at least twelve tonnes of ore" gives

$$x + y \geq 12$$

'Costs must be kept to less than $600/day ' gives

$$40x + 20y \leq 600$$

"Alberta must be less than twice the amount from Buffalo." gives

$$x \leq 2y$$

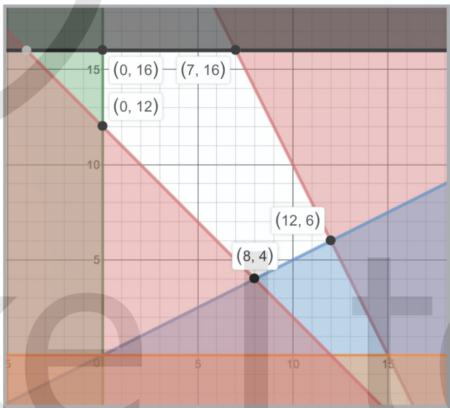
"Both are limited to 16 tonnes/day" gives

$$x \leq 16, \; y \leq 16$$

and we cannot have negative production gives

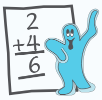
$$x \geq 0 \text{ and } y \geq 0$$

In entering these into Desmos we reverse the direction of all inequalities (shade out).



(0, 12)	prod.	$= 100 \times 0 + 150 \times 12$
		$= 1800$ kg/day
(12, 6)	prod.	$= 100 \times 12 + 150 \times 6$
		$= 2100$ kg/day
(7, 16)	prod.	$= 100 \times 7 + 150 \times 16$
		$= 3100$ kg/day MAXIMUM
(0, 16)	prod.	$= 100 \times 0 + 150 \times 16$
		$= 2400$ kg/day
(8, 4)	prod.	$= 100 \times 8 + 150 \times 4$
		$= 1400$ kg/day

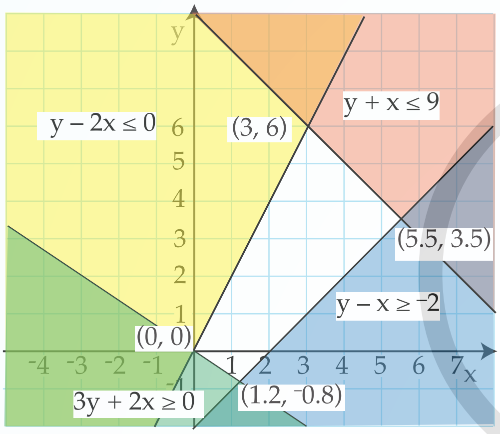
Should process 7 tonnes from mine Alberta and 16 tonnes from mine Buffalo each day.



Example

The graph shows the region defined by

$$y + x \leq 9$$
$$y - 2x \leq 0$$
$$y - x \geq {}^{-}2$$
$$3y + 2x \geq 0$$



a) Identify the maximum value function of F and when this occurs,

where $F = 5000 + 1500y - 1000x$

b) The function has now been changed to

$F = 5000 + 1500y - 1000kx$

For what negative value of k will there be multiple solutions including the point (3, 6)?

a) Checking each vertex for the value of F

$F = 5000 + 1500y - 1000x$

at (0, 0)	F = 5000	
at (1.2, $^{-}$0.8)	F = 2600	
at (3, 6)	P = 11 000	**Maximum**
at (5.5, 3.5)	P = 4750	

b) The gradient of $F = 5000 + 1500y - 1000kx$ is

$$m = \frac{1000k}{1500}$$

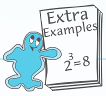
As k is negative we examine the negative gradient through (3, 6) which is

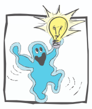
$$m = \frac{(3.5 - 6)}{(5.5 - 3)}$$

$$= {}^{-}1$$

Equating the two expression for gradient gives

$$k = {}^{-}1.5$$





You can check that your answer does give multiple solutions by checking both the vertices involved give the same maximum value for F when k = $^{-}$1.5

$F = 5000 + 1500y + 1500x$

at (0, 0)	**F = 5000**	**Minimum**
at (1.2, $^{-}$0.8)	**F = 5600**	
at (3, 6)	**P = 18 500**	**Maximum**
at (5.5, 3.5)	**P = 18 500**	**Maximum**

Therefore all the points from (3, 6) to (5.5, 3.5) will give the same maximum value for F.

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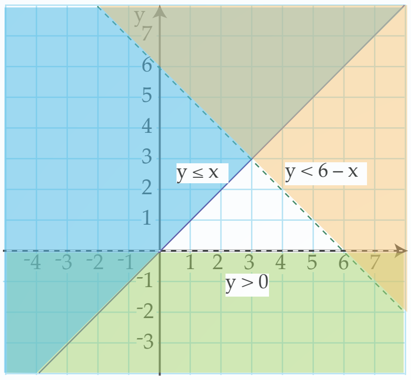
9.



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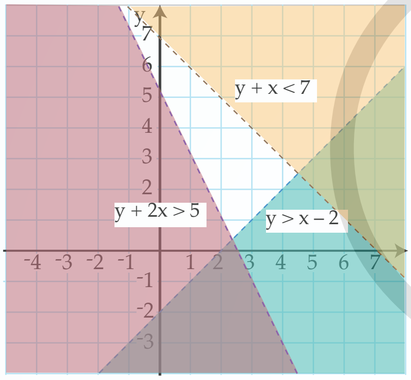

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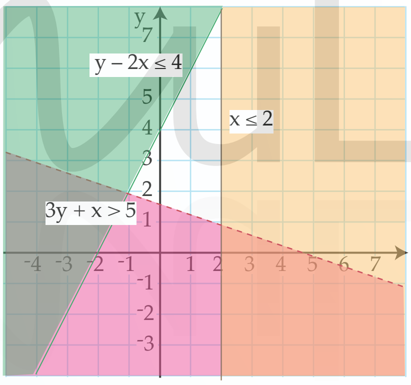
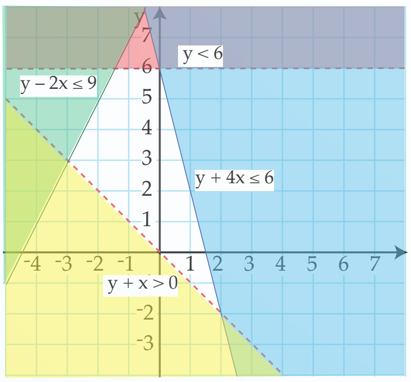
11. (2, 8), (2, 1) and ($^-$1, 2)



12. (2, $^-$2), (0, 6), ($^-$1.5, 6) and ($^-$3, 3)



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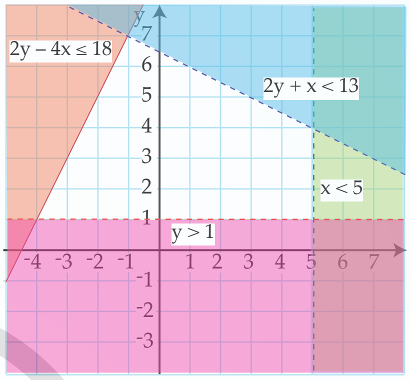
13. (5, 1), (5, 4), ($^-$1, 7) and ($^-$4, 1)



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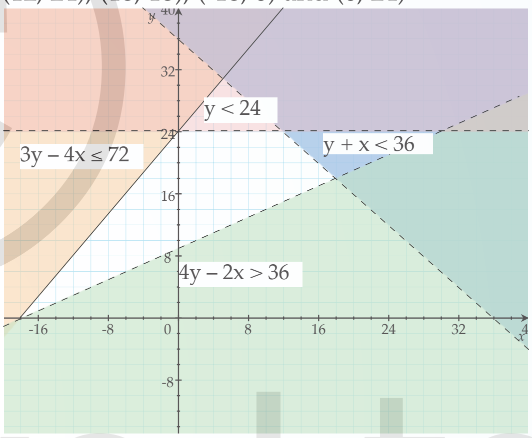
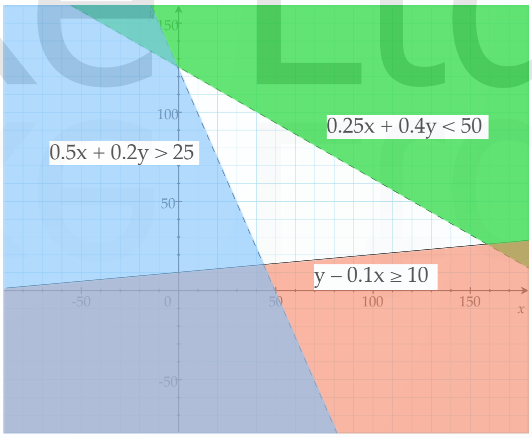
14. (12, 24), (18, 18), ($^-$18, 0) and (0, 24)



15. (158.6, 25.8), (0, 125) and (44.2, 14.4)



16. (0.5, 0), (0.5, 0.533), ($^-$0.12, 0.12) and (0.375, $^-$0.375)